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Vortices in the Two-Dimensional Simple Exclusion Process

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Abstract We show that the fluctuations of the partial current in two dimensional diffusive systems are dominated by vortices leading to a different scaling from the one predicted by the hydrodynamic large deviation theory. This is supported by exact computations of the variance of partial current fluctuations for the symmetric simple exclusion process on general graphs. On a two-dimensional torus, our exact expressions are compared to the results of numerical simulations. They confirm the logarithmic dependence on the system size of the fluctuations of the partial flux. The impact of the vortices on the validity of the fluctuation relation for partial currents is also discussed in an Appendix.

Keywords Exclusion processes · Diffusive systems · Partial currents · Fluctuation theorem · Vortices

1 Introduction

Recently, it has been shown how to compute the large deviation function of the current in one dimensional diffusive systems [2–7]. The hydrodynamic large deviation theory [2, 11, 18], yields explicit expressions for the large deviation function as well as the cumulants of the current fluctuations (under some stability condition [5, 7]). The same hydrodynamical approach applies in principle also to currents in higher dimension. In the present paper

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we show however that this approach does not always catch the correct scaling of the large deviations or of the cumulants of the current in higher dimensions. This will be made explicit in the case of the 2 dimensional symmetric simple exclusion process (SSEP).

For a *one dimensional* diffusive system of length L in contact at its left end with a reservoir at density ρ_a and at its right end with a reservoir at density ρ_b, one can consider the total net number $Q(\tau)$ of particles leaving the left reservoir during a time interval τ. This number $Q(\tau)$ fluctuates in time and one expects that in the long time limit

$$\mathrm{Pro}\left(\frac{Q(\tau)}{\tau} \simeq J\right) \sim \exp[-\tau G_L(J;\rho_a,\rho_b)] \tag{1.1}$$

where $G_L(J;\rho_a,\rho_b)$ is the large deviation function of the flux through the system. In fact G_L does not depend on where the flux, i.e. the integrated current, is measured along the one dimensional system, as long as particles cannot accumulate. For large L and for J of order $\frac{1}{L}$, G_L satisfies, the following scaling

$$G_L(J;\rho_a,\rho_b) \simeq \frac{1}{L} F(LJ;\rho_a,\rho_b). \tag{1.2}$$

The scaling (1.2) implies that for large L all the cumulants of $Q(\tau)$ are of order $1/L$, i.e.

$$\lim_{\tau\to\infty} \frac{\langle Q(\tau)^n\rangle_c}{\tau} \simeq \frac{1}{L}\kappa_n(\rho_a,\rho_b). \tag{1.3}$$

Explicit expressions of the $\kappa_n(\rho_a,\rho_b)$ have been obtained [5, 7] in terms of the diffusion constant $D(\rho)$ and the conductivity $\sigma(\rho)$ [22]. One can also show that the large deviation function G_L of the current satisfies the fluctuation theorem [2, 5, 7, 12–15, 19–21], i.e.

$$G_L(J;\rho_a,\rho_b) - G_L(-J;\rho_a,\rho_b) = J[\log z(\rho_b) - \log z(\rho_a)] \tag{1.4}$$

where $z(\rho)$ is the fugacity of a reservoir at density ρ.

In *higher dimension*, one can study, as in one dimension, the total current flowing through the system from one reservoir to the other, but one can also study part of this current. In this paper, we consider the SSEP on a square lattice of size L, with periodic boundary conditions in the vertical direction and study the current flowing through a vertical slit of length $\ell < L$ (see Fig. 1). The large deviation function $G_{L,\ell}(J;\rho_a,\rho_b)$, defined as in (1.1), depends of course on the size ℓ of the slit. One reason for considering the fluctuations of this partial

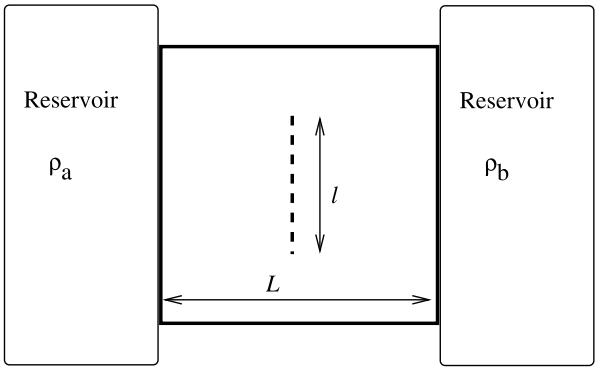

Fig. 1 We are going to consider the distribution of the current flowing through the dashed vertical slit of length $\ell < L$

current is that in experiments it is often only possible to measure the fluctuations of local quantities and not of global quantities [8, 9].

In two dimensions, when $\ell = L$, i.e. when one considers the total current flowing through the system, the large deviation function derived from the hydrodynamic theory satisfies for large L and for J of order 1 a scaling similar to the one dimensional case [4]

$$G_{L,L}(J;\rho_a,\rho_b) \simeq F(J;\rho_a,\rho_b) \tag{1.5}$$

(this would become $L^{d-2}F(L^{2-d}J;\rho_a,\rho_b)$ for a cube of size L in dimension d and J of order L^{d-2}). In the present paper we show that $G_{L,\ell}$ cannot satisfy the same scaling (1.5) as $G_{L,L}$ and that for large L, if one keeps the ratio $h = \ell/L$ fixed, then for all $0 < h < 1$ and J of order 1

$$G_{L,Lh}(J;\rho_a,\rho_b) \to 0 \quad \text{as } L \to \infty. \tag{1.6}$$

While, as in (1.3), one expects the cumulants of the total flux $Q(\tau)$ to have a large L limit

$$\lim_{\tau\to\infty} \frac{\langle Q(\tau)^n\rangle_c}{\tau} \to \kappa_n(\rho_a,\rho_b) \tag{1.7}$$

(which would become $\frac{1}{\tau}\langle Q(\tau)^n\rangle_c \simeq L^{d-2}\kappa_n(\rho_a,\rho_b)$ in dimension d), we will see by an explicit calculation of the second cumulant that for $\ell = Lh$,

$$\lim_{\tau\to\infty} \frac{\langle Q^{(h)}(\tau)^2\rangle_c}{\tau} \sim \log L \quad \text{as } L \to \infty, \text{ when } 0 < h < 1, \tag{1.8}$$

where $Q^{(h)}(\tau)$ is the flux of particles through the slit during time τ.

The fluctuation theorem, which is satisfied as written in (1.4) for the two-dimensional SSEP when J is the total current through the system (i.e. when $\ell = L$), has in fact no reason to remain valid for $\ell < L$: in the large L limit, the difference $G_{L,Lh}(J) - G_{L,Lh}(-J)$ vanishes when $0 < h < 1$ so that (1.4) cannot hold and a singular dependence can be expected in $G_{L,Lh}(J)$ when $h \to 1$. In Appendix A, we give a simple example of a two site model where one can see clearly that the fluctuation theorem is satisfied when one looks at the total current but is no longer valid when one considers only part of the current (see [1] for a discussion on the validity of the fluctuation theorem for partial currents).

The rest of the paper is organized as follows. In Sect. 2, we recall the hydrodynamic large deviation theory [2, 6] and show the asymptotics (1.6). Although the hydrodynamic large deviation theory does not predict the correct scaling of the current deviation, the analysis of Sect. 2 suggests that local current fluctuations are dominated by vortices. Restarting at the microscopic level, the variance of the integrated current is computed for the SSEP on a general graph (Sect. 3) and explicit expressions are obtained for the current through a slit for the SSEP on a two-dimensional torus (Sect. 4). Our exact expression leads to the asymptotics of the form (1.8) and are compared to the results of numerical simulations. Finally the appendices are devoted to comments on the fluctuation relation (1.4) for partial currents, and to some technical calculations. We note that Sects. 2, 3 and Appendix A can be read independently.

2 Vortices and Current Fluctuations

For simplicity, we briefly recall the large deviation hydrodynamic limit theory in the framework of the two-dimensional SSEP on the square lattice in the periodic domain $\Lambda = \{1, L\}^2$.

At the microscopic level, each particle jumps randomly with rate 1 to a nearest neighboring site and the jump is allowed only if the neighboring site is empty. After rescaling space by $1/L$ and time by $1/L^2$, the macroscopic density $\rho(x,t)$ obeys the diffusion equation [17, 22] in the macroscopic domain $\widehat{\Lambda} = [0,1]^2$, (with periodic boundary conditions),

$$\partial_t \rho(x,t) = \Delta\rho(x,t), \quad x = (x_1, x_2) \in \widehat{\Lambda}, \tag{2.1}$$

where Δ denotes the Laplacian. One can also define a macroscopic current $j(x,t) = (j_1(x,t), j_2(x,t))$ in the directions $\vec{e}_1, \vec{e}_2$ which has to satisfy

$$\partial_t \rho(x,t) = -\nabla \cdot j(x,t).$$

The rescaled current j is such that if $q_{(i,i+\vec{e}_\alpha)}(\tau)$ is the microscopic integrated current through the bond $(i, i+\vec{e}_\alpha)$ (with $\alpha = 1$ or 2) over the microscopic time interval $[0,\tau]$, then for a system of size L and times τ of order L^2, one has $q_{(i,i+\vec{e}_\alpha)}(\tau) = L\int_0^{\tau/L^2} j_\alpha(\frac{i}{L}, t)\,dt$.

Using the hydrodynamic large deviation theory, we are going to show that the scaling of the large deviations is different for the current flowing through the whole system or through a slit (as in Fig. 1).

2.1 Total Current Deviations

We denote by $Q(\tau)$ the integrated total current during the microscopic time interval $[0,\tau]$ through a vertical section of the whole system, say the current flowing through the edges $\{(L/2, i_2), (L/2+1, i_2)\}_{1\le i_2\le L}$. The corresponding large deviation function $G_{L,L}$ is defined by

$$\lim_{\tau\to\infty} -\frac{1}{\tau}\log \mathrm{Pro}\left(\frac{Q(\tau)}{\tau} \approx J\right) = G_{L,L}(J), \tag{2.2}$$

where $\mathrm{Pro}(\frac{Q(\tau)}{\tau} \approx J)$ denotes the probability of observing a total current J in the $\vec{e}_1$ direction averaged over the microscopic time interval $[0,\tau]$. According to the large deviation hydrodynamic theory, one expects, in accord with (1.5) that $\lim_{L\to\infty} G_{L,L}(J) = F(J)$ where the function $F(J) = \lim_{T\to\infty} F_T(J)$ with

$$\begin{aligned} F_T(J) &= \inf_{j,\rho}\left\{\frac{1}{T}\mathcal{I}_T(j,\rho)\right\}, \quad \text{and} \\ \mathcal{I}_T(j,\rho) &= \frac{1}{2}\int_0^T\int_{\widehat{\Lambda}} dt\,dx\,\frac{\left(j_1+\partial_{x_1}\rho\right)^2 + \left(j_2+\partial_{x_2}\rho\right)^2}{2\rho(1-\rho)}. \end{aligned} \tag{2.3}$$

The minimum is taken over the macroscopic evolutions $\{\rho(x,t), j(x,t)\}$ during the macroscopic time interval $[0,T]$ which satisfy the constraints

$$\partial_t\rho(x,t) = -\nabla\cdot j(x,t), \quad \text{and} \quad J = \frac{1}{T}\int_0^T\int_0^1 dt\,dx_2\,j_1\left(\left(\frac{1}{2}, x_2\right), t\right). \tag{2.4}$$

Remark 2.1 Note that the mathematical statement from the hydrodynamic limit theory [3] relies on a more involved asymptotic with a joint space/time scaling: instead of (2.2), the large deviation function for a total current J over the microscopic time interval $[0, L^2T]$ is given by

$$\lim_{L\to\infty} -\frac{1}{L^2T}\log \mathrm{Pro}\left(\frac{Q(L^2T)}{L^2T} \approx J\right) = F_T(J),$$

where F_T has been introduced in (2.3). When writing (2.2), (2.3), we assumed that in the previous expression the limits $L \to \infty$ and $T \to \infty$ can be exchanged.

As we consider in this section a system with periodic boundary conditions and no sources, the steady state is the equilibrium one in which all configurations with a specified total of number particles have equal weight. The mean current through the system is therefore 0 and we are going to show that for any current deviation $J \neq 0$

$$F(J) > 0. \tag{2.5}$$

Expanding $\mathcal{I}_T$ in (2.3) and using Jensen's inequality leads to

$$\begin{aligned}\mathcal{I}_T(j,\rho) &= \frac{1}{2}\int_0^T\int_{\widehat{\Lambda}} dt\,dx\left[\frac{(j_1)^2+(j_2)^2}{2\rho(1-\rho)}+\frac{\left(\nabla\rho\right)^2}{2\rho(1-\rho)}\right]+C_T\\ &\geq \int_0^T\int_{\widehat{\Lambda}} dt\,dx(j_1)^2+C_T \geq T\left(\frac{1}{T}\int_0^T\int_{\widehat{\Lambda}} dt\,dx j_1\right)^2+C_T,\end{aligned} \tag{2.6}$$

where C_T is the contribution of the cross terms in (2.3)

$$C_T = \frac{1}{2}\int_0^T\int_{\widehat{\Lambda}} dt\,dx\frac{j\cdot\nabla\rho}{\rho(1-\rho)} = \frac{1}{2}\int_{\widehat{\Lambda}} dx\{\mathcal{S}(\rho(x,0))-\mathcal{S}(\rho(x,T))\}$$

with $\mathcal{S}(\rho) = -[\rho\log(\rho)+(1-\rho)\log(1-\rho)]$. As it is equivalent to measure the total current through any section of the system, the constraint (2.4) on the current deviations becomes

$$J = \frac{1}{T}\int_0^T\int_{\widehat{\Lambda}} dt\,dx j_1(x,t). \tag{2.7}$$

Thus $\inf_{j,\rho}\{\mathcal{I}_T(j,\rho)\} \geq TJ^2 + C_T$. As C_T remains bounded in time, (2.5) follows from (2.3).

2.2 Partial Current Deviations

The functional $\mathcal{I}_T$ defined in (2.3) should in principle provide the large deviations of the current through any macroscopic region of the system. We consider now a slit of macroscopic height $h < 1$ (the segment $[(1/2,0),(1/2,h)]$) and denote by $Q^{(h)}(\tau)$ the integrated current through the slit during the microscopic time interval $[0,\tau]$, i.e. the current flowing through the edges $\{(L/2,i_2),(L/2+1,i_2)\}_{1\leq i_2\leq hL}$. Then, the large deviation function for observing a current deviation $J \neq 0$ is given by

$$\lim_{\tau\to\infty} -\frac{1}{\tau}\log\mathrm{Pro}\left(\frac{Q^{(h)}(\tau)}{\tau}\approx J\right) = G_{L,Lh}(J).$$

One expects from (1.5), that $\lim_{L\to\infty} G_{L,Lh}(J) = F_h(J)$ with

$$F_h(J) = \lim_{T\to\infty}\inf_{j,\rho}\left\{\frac{1}{T}\mathcal{I}_T(j,\rho)\right\} \tag{2.8}$$

where $\mathcal{I}_T(j,\rho)$ is defined in (2.3) and the macroscopic constraints (2.4) are replaced by

$$\partial_t\rho(x,t) = -\nabla\cdot j(x,t), \quad \text{and} \quad J = \frac{1}{T}\int_0^T\int_0^h dt\,dx_2 j_1\left(\left(\frac{1}{2},x_2\right),t\right). \tag{2.9}$$

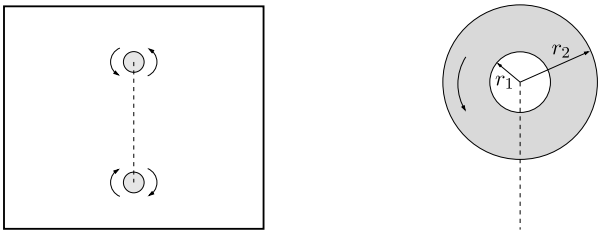

Fig. 2 *On the left*, two vortices located at the edges of the dashed slit are depicted. *On the right*, a blow up of one vortex concentrated on the disk of radius (r_1, r_2): the current (2.12), (2.13) is proportional to $1/r$ for r in (r_1, r_2) and vanishes outside this annulus

We are going to show that in contrast to (2.5), the large deviation function F_h in (2.8) vanishes for $0 < h < 1$ (as claimed in (1.6)).

One can bound (2.8) by

$$\inf_{j,\rho}\left\{\frac{1}{T}\,\mathcal{I}_T(j,\rho)\right\} \le \tilde{F}_h(J),$$

where the functional $\tilde{F}_h(J)$ is the restriction of $\mathcal{I}_T$ to time independent density and current profiles,

$$\tilde{F}_h(J) = \inf_{j,\rho}\left\{\frac{1}{2}\int_{\widehat{\Lambda}} dx\left[\frac{(j_1)^2+(j_2)^2}{2\rho(1-\rho)} + \frac{(\nabla\rho)^2}{2\rho(1-\rho)}\right]\right\} \tag{2.10}$$

where the density and current constraints satisfy

$$0 = \nabla\cdot j = \partial_1 j_1(x) + \partial_2 j_2(x), \quad \text{and} \quad J = \int_0^h dx_2\, j_1\left(\frac{1}{2}, x_2\right). \tag{2.11}$$

A guess to bound (2.10) is to consider the equilibrium density (uniformly equal to the constant density $\bar{\rho}$) and a current deviation consisting of two vortices localized at the edges of the slit $(1/2, 0)$ and $(1/2, h)$ (see Fig. 2)

$$\forall x \in \widehat{\Lambda}, \quad j(x) = J(\Phi(x-(1/2,h)) - \Phi(x-(1/2,0))), \tag{2.12}$$

where Φ denotes the vector

$$\forall x = (x_1, x_2), \quad \Phi(x) = \frac{1}{2\log(r_2/r_1)}\,\frac{1_{\{r_1 \le \sqrt{x_1^2+x_2^2} \le r_2\}}}{x_1^2+x_2^2}(-x_2, x_1), \tag{2.13}$$

with $r_1 < r_2 \ll 1$. One can check that the current defined in (2.12) satisfies the constraint (2.11). Furthermore, we can bound $\tilde{F}_h(J)$ by,

$$\tilde{F}_h(J) \le \frac{1}{4\bar{\rho}(1-\bar{\rho})}\int_{\widehat{\Lambda}} dx[(j_1)^2+(j_2)^2] = \frac{\pi}{4\bar{\rho}(1-\bar{\rho})}\,\frac{J^2}{\log(r_2/r_1)}. \tag{2.14}$$

Letting r_1, r_2 go to 0 while $\frac{r_2}{r_1}$ diverges, we find that $\tilde{F}_h(J) = 0$ so that the large deviation cost in (2.8) is 0.

Remark 2.2 On a finite lattice of size L, the current has to flow through the bonds and therefore the ratio r_2/r_1 is at most L. This imposes a cut-off and the computation (2.14) based on (2.10) leads to $G_{L,Lh}(J)$ of order $\frac{1}{\log L}$. This logarithmic dependence will be confirmed for the SSEP by a direct computation of the current fluctuations in Sects. 3 and 4.

Expression (2.14) shows that the cost of the fluctuations due to the vortices is low and one may wonder if the vortices are the optimal minimizers of (2.10) or whether one should expect a more complex structure. To check this, we first note that the current j in (2.11) is divergence free, thus it can be represented as the sum of the curl of a vector field $(0, 0, \Psi(x_1, x_2))$ and a constant vector field (C_1, C_2)

$$j = \nabla \times (0, 0, \Psi) + (C_1, C_2) = (\partial_2 \Psi, -\partial_1 \Psi) + (C_1, C_2) \tag{2.15}$$

and the current constraint (2.11) becomes

$$J = \Psi(1/2, h) - \Psi(1/2, 0) + C_1 h. \tag{2.16}$$

Choosing the density equal to $\bar{\rho}$, (2.10) reduces to the variational principle

$$\tilde{F}_h(J) \leq \frac{1}{4\bar{\rho}(1-\bar{\rho})} \inf_{\Psi,(C_1,C_2)} \left\{ \int_{\widehat{\Lambda}} dx (\partial_1 \Psi)^2 + (\partial_2 \Psi)^2 + C_1^2 + C_2^2 \right\} \tag{2.17}$$

where Ψ and C_1 satisfy the constraint (2.16). The solutions of the above variational problem will satisfy

$$\Delta \Psi(x) = \alpha(\delta_{x,(1/2,h)} - \delta_{x,(1/2,0)}),$$

where α is the Lagrange parameter associated to the constraint (2.16). This would lead to a Ψ which diverges logarithmically at the edges of the slit and therefore cannot satisfy condition (2.16) for any non-zero α. Nevertheless, using a cut-off similar to r_1 in (2.13), we can recover the vortex like structures (2.12).

Remark 2.3 For more general diffusive systems the hydrodynamic large deviations are governed by functionals of the type (2.3) which depend on diffusion and conductivity matrices [2]. One could extend the previous discussion to these cases and the large deviation function $F_h(J)$ would vanish as soon as $h < 1$. For open systems, similar computations can be done as the fluctuations are dominated by vortices localized at the edges of the slit (2.12) and the reservoirs play a negligible role.

Remark 2.4 We note that in analogy to (2.7) we can consider the partial current specified in (2.9) as a limiting case of an integrated current in a domain $B \subset \widehat{\Lambda}$,

$$J_B = \frac{1}{T} \int_0^T \int_B j_1(x, t) dx. \tag{2.18}$$

Taking B to be a rectangle of height h and width w we get that the flux through the line segment h is given by $w^{-1} J_B$, in the limit $w \to 0$. For the large deviation of J_B one can repeat the analysis leading to (2.6) yielding

$$\inf_{j,\rho} \mathcal{I}_T(j, \rho : B) \geq T \frac{J_B^2}{|B|} + C_T. \tag{2.19}$$

This non-zero lower bound reflects the fact that any vortex flow used to implement the fluctuation J_B will have to be of a size w or greater.

3 Current Fluctuations on a General Graph

In this section, we consider the SSEP on a general connected graph $(\Lambda, \mathcal{E}_\Lambda)$ where Λ is a finite set of sites and $\mathcal{E}_\Lambda$ the set of edges on which particles jump with rate 1 according to the exclusion rule. We also suppose that particles are created and annihilated at the sites in the subset Ω of Λ (Ω may be empty). For any site i in Ω, we suppose that creation and annihilation occur at rates α_i and β_i (and for simplicity we choose $\alpha_i + \beta_i = 1$). In Sect. 4, we will apply the results obtained for general graphs to the microscopic domain $\Lambda = \{1, L\}^2$ and derive explicit expressions in this case.

If (i, j) is an edge in $\mathcal{E}_\Lambda$ then the number of particle jumps from i to j during the time interval τ is denoted by $q_{(i,j)}(\tau)$. The current flowing through (i, j) during time τ is then $q_{(i,j)}(\tau) - q_{(j,i)}(\tau)$. If creation and annihilation occur, we enlarge the graph $(\Lambda, \mathcal{E}_\Lambda)$ by associating to each site i in Ω a new site $\bar{i}$. The site $\bar{i}$ can be interpreted as a source and we denote by $q_{(\bar{i},i)}(\tau)$ the number of particles created at i and $q_{(i,\bar{i})}(\tau)$ the number of particles annihilated at site i. It is convenient to enlarge the graph $\mathcal{E}_\Lambda$ into $\bar{\mathcal{E}}_\Lambda$ by adding to the original graph the new edges $(i, \bar{i})_{i\in\Omega}$ and $(\bar{i}, i)_{i\in\Omega}$. We denote by $\bar{\Omega}$ the set of the new sites, and by $\bar{\Lambda}$ the union of Λ and $\bar{\Omega}$.

For any field $\{\mathcal{A}_b\}_b$ indexed by the edges and such that for any edge (i, j), $\mathcal{A}_{(i,j)} = -\mathcal{A}_{(j,i)}$, we are going to study the fluctuation of the integrated current defined by

$$\mathbb{Q}^{\mathcal{A}}(\tau) = \sum_{b\in\bar{\mathcal{E}}_\Lambda} \mathcal{A}_b\, q_b(\tau), \tag{3.1}$$

where the sum is over all the oriented edges b. The field $\mathcal{A}_b$ can be thought as a test function.

One can define the divergence and the gradient on the graph. For any field $\{\mathcal{A}_b\}_b$ and any site i in Λ

$$\operatorname{div} \mathcal{A}(i) = \sum_{j\sim i} \mathcal{A}_{(i,j)}, \tag{3.2}$$

where the sum is over all the edges leaving site i (this includes the edges $(i, \bar{i})$ if creation or annihilation occur at site i). For any function H_i in $\bar{\Lambda}$, the gradient is a function indexed by the edges $b = (i, j)$ in $\bar{\mathcal{E}}_\Lambda$

$$\nabla_b H = H_j - H_i. \tag{3.3}$$

In the following, we will consider only functions H in $\bar{\Lambda}$ equal to 0 in $\bar{\Omega}$.

3.1 Gauge Invariance

Before, computing the variance of $\mathbb{Q}^{\mathcal{A}}(\tau)$ defined in (3.1), we first show that for large τ similar asymptotics of $\log\langle\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle$ can be obtained for different choices of $\{\mathcal{A}_b\}_b$.

For any site i in Λ, one has with notation (3.2)

$$\eta(i,\tau) - \eta(i,0) = \sum_{\substack{j\in\bar{\Lambda},\\ j\sim i}} q_{(j,i)}(\tau) - q_{(i,j)}(\tau). \tag{3.4}$$

This implies that for any function H on $\bar{\Lambda}$ (equal to 0 on $\bar{\Omega}$)

$$\begin{aligned}\mathbb{Q}^{\nabla H}(\tau) &= \sum_{b\in\bar{\mathcal{E}}_\Lambda} \nabla_b H\, q_b(\tau) = \sum_{i\in\bar{\Lambda}} H_i \Bigg(\sum_{\substack{j\in\bar{\Lambda},\\ j\sim i}} q_{(j,i)}(\tau) - q_{(i,j)}(\tau) \Bigg) \\ &= \sum_{i\in\Lambda} H_i(\eta(i,\tau) - \eta(i,0)).\end{aligned} \tag{3.5}$$

Therefore, $\mathbb{Q}^{\nabla H}(\tau)$ remains bounded when the time diverges. As a consequence, for any $\lambda \in \mathbb{R}$, $\mathcal{A}_b$ and H_i

$$\lim_{\tau\to\infty} \frac{1}{\tau} \log\left\langle \exp\left(\lambda \sum_{b\in\bar{\mathcal{E}}_\Lambda} \mathcal{A}_b\, q_b(\tau)\right)\right\rangle$$
$$= \lim_{\tau\to\infty} \frac{1}{\tau} \log\left\langle \exp\left(\lambda \sum_{b\in\bar{\mathcal{E}}_\Lambda} (\mathcal{A}_b + \nabla_b H) q_b(\tau)\right)\right\rangle. \tag{3.6}$$

Thus the large deviations of $\mathbb{Q}^{\mathcal{A}}(\tau)$ and $\mathbb{Q}^{\mathcal{A}+\nabla H}(\tau)$ with respect to time are the same.

We are going now to recall how a field $\mathcal{A}_b$ in $\bar{\mathcal{E}}_\Lambda$ can be decomposed as

$$\mathcal{A}_b = V_b + \nabla_b H, \tag{3.7}$$

where H is a function in $\bar{\Lambda}$ (equal to 0 on $\bar{\Omega}$) and V is divergence free in Λ

$$\forall i \in \Lambda, \quad \operatorname{div} V(i) = 0.$$

Note that no conditions are imposed on $\operatorname{div} V(\bar{i})$ for $\bar{i}$ in $\bar{\Omega}$, if there are sources ($\Omega \neq \emptyset$). If there are no sources ($\Omega = \emptyset$), then H is defined up to a constant.

For decomposition (3.7) to hold, H has to be the solution of

$$\forall i \in \Lambda, \quad \operatorname{div} \mathcal{A}(i) = \Delta H_i = \sum_{j\sim i} (H_j - H_i). \tag{3.8}$$

In the case with sources ($\Omega \neq \emptyset$), the solution of (3.8) can be written in terms of the Green's functions, defined for any site k by

$$\forall i \in \Lambda, \quad \Delta G_i^{(k)} = -\delta_{i,k}, \quad \text{and} \quad \forall i \in \bar{\Omega}, \quad G_i^{(k)} = 0. \tag{3.9}$$

Thus for $j \in \bar{\Lambda}$

$$H_j = -\sum_{k\in\Lambda} \operatorname{div} \mathcal{A}(k) G_j^{(k)}, \tag{3.10}$$

and the field $V_b = \mathcal{A}_b - \nabla_b H$ is divergence free.

3.2 Variance of the Current

We are going to compute the variance of $\mathbb{Q}^{\mathcal{A}}(\tau) = \sum_{b\in\mathcal{E}_\Lambda} \mathcal{A}_b\, q_b(\tau)$. From (3.6–3.7), we know that to compute large time asymptotics it is enough to consider $\mathcal{A}$ which is divergence free.

One has

$$\partial_\tau \log\langle \exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle$$
$$= \sum_{i\in\Omega} \alpha_i (\exp(\lambda \mathcal{A}_{(\bar{i},i)}) - 1) \frac{\langle (1-\eta_i) \exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle}{\langle \exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle}$$

$$+ \beta_i(\exp(-\lambda\mathcal{A}_{(\bar{i},i)}) - 1)\frac{\langle \eta_i \exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle}{\langle \exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle}$$
$$+ \sum_{(i,j)\in\mathcal{E}_\Lambda} (\exp(\lambda\mathcal{A}_{(i,j)}) - 1)\frac{\langle \eta_i(1-\eta_j)\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle}{\langle \exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle}, \tag{3.11}$$

where the sum is over all the oriented bonds (i, j) and $\langle\cdot\rangle$ denotes the average over the random process in the time interval $[0, \tau]$ and over an initial condition chosen according to the invariant measure for the SSEP. The procedure to derive (3.11) is similar to what was done in [10]. One considers all the possible moves occurring during an infinitesimal time interval $d\tau$ and their contributions to $\langle \exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle$. The first terms in (3.11) correspond to a jump of a particle from site $\bar{i}$ to site i (creation) or from site i to site $\bar{i}$ (annihilation), whereas the last term corresponds to a jump from site i to j.

Let us denote by $\langle\cdot\rangle_\lambda$ the expectation of the tilted measure defined as follows: for any function f

$$\langle f(\eta)\rangle_\lambda = \lim_{\tau\to\infty} \frac{\langle f(\eta(\tau))\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle}{\langle \exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle}. \tag{3.12}$$

Using the symmetry $\mathcal{A}_{(i,j)} = -\mathcal{A}_{(j,i)}$ and the relation $\alpha_i + \beta_i = 1$, we get by expanding (3.11) for small λ

$$\begin{aligned}
&\lim_{\tau\to\infty} \partial_\tau \log\langle \exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle \\
&\quad = \lambda\sum_{i\in\Omega}\mathcal{A}_{(\bar{i},i)}\alpha_i + \lambda\sum_{i\in\Lambda}\operatorname{div}\mathcal{A}(i)\langle\eta_i\rangle_\lambda + \frac{\lambda^2}{2}\sum_{i\in\Omega}(\mathcal{A}_{(\bar{i},i)})^2\langle(\beta_i\eta_i + \alpha_i(1-\eta_i))\rangle_\lambda \\
&\qquad + \frac{\lambda^2}{2}\sum_{(i,j)\in\mathcal{E}_\Lambda}(\mathcal{A}_{(i,j)})^2\langle\eta_i(1-\eta_j)\rangle_\lambda + O(\lambda^3).
\end{aligned} \tag{3.13}$$

For λ small, one expects that

$$\langle\eta_i\rangle_\lambda = \langle\eta_i\rangle + O(\lambda), \qquad \langle\eta_i(1-\eta_j)\rangle_\lambda = \langle\eta_i(1-\eta_j)\rangle + O(\lambda). \tag{3.14}$$

In principle one would need to know the first order correction to $\langle\eta_i\rangle_\lambda$ to obtain (3.11) at the second order in λ.

For $\mathcal{A}$ divergence free, the term $\langle\eta_i\rangle_\lambda$ disappears and the formula (3.13) simplifies

$$\lim_{\tau\to\infty}\frac{1}{\tau}\log\langle\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle = \lambda\,\mathrm{Mean} + \frac{\lambda^2}{2}\mathrm{Var} + O(\lambda^3), \tag{3.15}$$

where

$$\mathrm{Mean} = \lim_{\tau\to\infty}\frac{\langle\mathbb{Q}^{\mathcal{A}}(\tau)\rangle}{\tau} = \sum_{i\in\Omega}\mathcal{A}_{(\bar{i},i)}\alpha_i.$$

$$\begin{aligned}
\mathrm{Var} &= \lim_{\tau\to\infty}\frac{\langle\mathbb{Q}^{\mathcal{A}}(\tau)^2\rangle_c}{\tau} \\
&= \sum_{(i,j)\in\mathcal{E}_\Lambda}(\mathcal{A}_{(i,j)})^2\langle\eta_i(1-\eta_j)\rangle + \sum_{i\in\Omega}(\mathcal{A}_{(\bar{i},i)})^2\langle\eta_i(1-\alpha_i) + \alpha_i(1-\eta_i)\rangle,
\end{aligned} \tag{3.16}$$

where the sum is over all the oriented edges. If there are no sources then the second term in (3.16) disappears.

For a general field $\mathcal{A}$, we can use the decomposition (3.7). If $\Omega \neq 0$, there is a representation of V in terms of Green functions (3.10): for any $(i, j) \in \bar{\Lambda}$

$$V_{(i,j)} = \mathcal{A}_{(i,j)} + \sum_{k\in\Lambda} \operatorname{div} \mathcal{A}(k)(-G_i^{(k)} + G_j^{(k)}). \tag{3.17}$$

Invariance (3.6) implies that the asymptotic formula for (3.17) is given by (3.15) with $\mathcal{A}$ replaced by V

$$\text{Mean} = \sum_{i\in\Omega} V_{(\tilde{i},i)}\alpha_i.$$

$$\text{Var} = \sum_{(i,j)\in\mathcal{E}_\Lambda} (V_{(i,j)})^2 \langle \eta_i(1-\eta_j)\rangle + \sum_{i\in\Omega} (V_{(\tilde{i},i)})^2 \langle \eta_i(1-\alpha_i) + \alpha_i(1-\eta_i)\rangle, \tag{3.18}$$

where as before the sum is over all the oriented edges.

Further simplifications can be obtained if the system is in equilibrium at density $\bar{\rho}$, i.e. if the intensities of the sources are such that $\alpha_i = \bar{\rho}$, $\beta_i = 1 - \bar{\rho}$. In this case, $\langle \eta_i(1-\eta_j)\rangle = \bar{\rho}(1-\bar{\rho})$ and by expanding (3.18) with $V_b = \mathcal{A}_b - \nabla_b H$, we get

$$\begin{aligned}\text{Var} &= \bar{\rho}(1-\bar{\rho})\Bigg[\sum_{(i,j)\in\bar{\mathcal{E}}_\Lambda} \mathcal{A}^2_{(i,j)} - 2\sum_{(i,j)\in\bar{\mathcal{E}}_\Lambda} \mathcal{A}_{(i,j)}\nabla_{(i,j)}H + \sum_{(i,j)\in\bar{\mathcal{E}}_\Lambda} (\nabla_{(i,j)}H)^2\Bigg] \\ &= \bar{\rho}(1-\bar{\rho})\Bigg[\sum_{(i,j)\in\bar{\mathcal{E}}_\Lambda} \mathcal{A}^2_{(i,j)} + 4\sum_{i\in\Lambda} \operatorname{div}\mathcal{A}(i)H_i - 2\sum_{i\in\Lambda} \Delta H_i H_i\Bigg],\end{aligned}$$

where the second equation is obtained by summation by parts. From (3.8), one has $\Delta H_i = \operatorname{div}\mathcal{A}(i)$ so that

$$\text{Var} = \bar{\rho}(1-\bar{\rho})\Bigg[\sum_{(i,j)\in\bar{\mathcal{E}}_\Lambda} \mathcal{A}^2_{(i,j)} + 2\sum_{i\in\Lambda} \operatorname{div}\mathcal{A}(i)H_i\Bigg]. \tag{3.19}$$

Replacing H by (3.10), we finally obtain

$$\text{Var} = \bar{\rho}(1-\bar{\rho})\Bigg[\sum_{(i,j)\in\bar{\mathcal{E}}_\Lambda} \mathcal{A}^2_{(i,j)} - 2\sum_{i,k\in\Lambda} G_i^{(k)} \operatorname{div}\mathcal{A}(k)\operatorname{div}\mathcal{A}(i)\Bigg], \tag{3.20}$$

where the sum is over all the oriented edges (i, j).

4 Two Dimensional SSEP

In this section we will apply the general results of Sect. 3 to the SSEP in the periodic square lattice $\Lambda = \{1, L\}^2$ with nearest neighbor jumps and derive explicit expressions in this case. We consider the integrated current flowing through the edges in $\Gamma_L^\ell = \{L/2, L/2+1\} \times \{1, \ell\}$ given by

$$Q^\ell(\tau) = \sum_{(i,i+\vec{e}_1)\in\Gamma_L^\ell} q_{(i,i+\vec{e}_1)}(\tau) - q_{(i+\vec{e}_1,i)}(\tau) \tag{4.1}$$

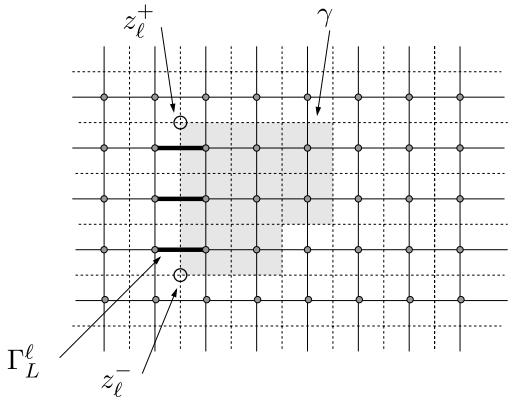

Fig. 3 The *dashed lines* represent the dual lattice and Γ_L^ℓ is depicted by the thick edges. The function H defined in (4.3) is equal to 1 in the *grey region* and 0 outside

with $\vec{e}_1 = (1, 0)$. The integrated current Q^ℓ can be rewritten as $\mathbb{Q}^{\mathcal{A}}$ defined in (3.1) with

$$\forall i, j \in \Lambda, \quad \mathcal{A}_{i,j} = \begin{cases} 1, & \text{if } (i, j) = (i, i + \vec{e}_1) \in \Gamma_L^\ell, \\ -1, & \text{if } (i, j) = (i, i - \vec{e}_1) \in \Gamma_L^\ell, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

The gauge invariance (3.6) is easily illustrated in the two dimensional case. Let z_ℓ^+ and z_ℓ^- be the 2 sites of the dual lattice such that Γ_L^ℓ is the set of edges intersected by the segment (z_ℓ^+, z_ℓ^-) (see Fig. 3). Let γ be another path connecting z_ℓ^+ to z_ℓ^- on the dual lattice, then we can define the current $\mathbb{Q}^{\mathcal{B}}(\tau)$ flowing through the edges intersecting γ, where $\mathcal{B}$ generalizes (4.2) for the edges intersecting γ. One can check that

$$\mathcal{A} = \mathcal{B} + \nabla H, \tag{4.3}$$

for some H. Therefore, the statistics of the currents $\mathbb{Q}^{\mathcal{A}}(\tau)$ and $\mathbb{Q}^{\mathcal{B}}(\tau)$ are asymptotically the same at large times (3.6). This is of course expected as the particles in the grey region of Fig. 3 cannot accumulate.

4.1 Computation of the Variance

We turn now to the computation of

$$\mathrm{Var} = \lim_{\tau \to \infty} \frac{\langle Q^\ell(\tau)^2 \rangle_c}{\tau},$$

the asymptotic of the variance of $Q^\ell(\tau) = \mathbb{Q}^{\mathcal{A}}(\tau)$ (4.2) for large τ. On the periodic domain, the variance is given by (3.18) without the source term. As the invariant measure is uniformly distributed, $\langle \eta_i(1 - \eta_j) \rangle$ depends only on the number N of particles and the size L. Let $S_{L,N} = \frac{N(L^2-N)}{L^2(L^2-1)} = \langle \eta_i(1 - \eta_j) \rangle$ for $i \neq j$, then the expression (3.19) remains valid

$$\mathrm{Var} = S_{L,N} \Big[\sum_{(i,j) \in \mathcal{E}_\Lambda} \mathcal{A}_{(i,j)}^2 + 2 \sum_{i \in \Lambda} \mathrm{div}\, \mathcal{A}(i) H_i \Big], \tag{4.4}$$

where H is given by (3.8) which reads now

$$\forall i \in \Lambda, \quad \Delta H_i = \operatorname{div} \mathcal{A}(i) = \begin{cases} 1, & \text{if } i \in \Gamma_L^{\ell,+} = \{L/2, s\}_{1 \le s \le \ell}, \\ -1, & \text{if } i \in \Gamma_L^{\ell,-} = \{L/2 + \vec{e}_1, s\}_{1 \le s \le \ell}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

Thus H is equal to

$$H_i = -\sum_{k \in \Gamma_L^{\ell,+}} G_i^{(k,k+\vec{e}_1)}, \tag{4.6}$$

where the Green's function (3.9) is replaced for any sites k, k' in Λ by

$$\forall i \in \Lambda, \quad \Delta G_i^{(k',k)} = \delta_{i,k} - \delta_{i,k'}. \tag{4.7}$$

From (4.4), we finally obtain

$$\mathrm{Var} = 2S_{L,N}\left[\ell - \sum_{i,k \in \Gamma_L^{\ell,+}} (G_i^{(k,k+\vec{e}_1)} - G_{i+\vec{e}_1}^{(k,k+\vec{e}_1)})\right]. \tag{4.8}$$

The Green's function (4.7) is given for any $i = (i_1, i_2)$ in Λ by

$$G_i^{(k,k')} = \frac{1}{L^2} \sum_{q_1,q_2 \neq (0,0)} \frac{\exp(\mathbf{i}q \cdot (i-k)) - \exp(\mathbf{i}q \cdot (i-k'))}{4 - 2\cos(q_1) - 2\cos(q_2)} \tag{4.9}$$

where $q \cdot j = q_1 j_1 + q_2 j_2$ stands for the scalar product with $q_1 = 2\pi \frac{m_1}{L}, q_2 = 2\pi \frac{m_2}{L}$ for m_1, m_2 in $\{0, L-1\}$. Thus (4.8) becomes with the convention that $\frac{1-\cos(q_2\ell)}{1-\cos(q_2)} = \ell^2$ for $q_2 = 0$

$$\mathrm{Var} = 2S_{L,N}\left(\ell - \frac{1}{L^2} \sum_{q_1,q_2 \neq (0,0)} \frac{(1-\cos(q_1))(1-\cos(q_2\ell))}{(1-\cos(q_2))(2-\cos(q_1)-\cos(q_2))}\right). \tag{4.10}$$

Using the identity

$$\frac{1}{L} \sum_{q_2} \frac{1-\cos(q_2\ell)}{1-\cos(q_2)} = \frac{1}{L} \sum_{m=1}^{L-1} \frac{1-\cos(2\pi\ell\frac{m}{L})}{1-\cos(2\pi\frac{m}{L})} + \frac{\ell^2}{L} = \ell,$$

we finally rewrite (4.10) as

$$\mathrm{Var} = 2S_{L,N}\left(\frac{\ell^2}{L^2} + \frac{1}{L^2} \sum_{q_1,q_2 \neq (0,0)} \frac{1-\cos(q_2\ell)}{2-\cos(q_1)-\cos(q_2)}\right) \tag{4.11}$$

with $q_1 = 2\pi \frac{m_1}{L}$, $q_2 = 2\pi \frac{m_2}{L}$ for m_1, m_2 in $\{0, L-1\}$.

One can show, see Appendix B, that for large L and ℓ, expression (4.11) becomes for $h = \ell/L$

$$\mathrm{Var} = \frac{2S_{L,N}}{\pi}\left[\log L + h^2 + \log\left(\frac{\sinh(\pi h)}{\pi}\right) + \frac{3\log 2}{2} + \gamma \right.$$
$$\left. + \sum_{m\geq 1} \log\left(1 + \frac{\sin^2(\pi h)}{\sinh^2(\pi m)}\right)\right], \tag{4.12}$$

where $\gamma \approx 0.577$ is the Euler constant.

4.2 Time Dependence

To compare with the results of simulations, it is useful to calculate how the moments of $\mathbb{Q}^{\mathcal{A}}(\tau)$ depend on τ. At finite times τ, the gauge invariance (3.6) is of no use. We focus now on systems with no sources and study the variance of $\mathbb{Q}^{\mathcal{A}}(\tau)$ for a general field $\mathcal{A}$ at finite time τ.

Following the same procedure which led to (3.11), (3.13), we get up to the second order in λ,

$$\begin{aligned}
&\partial_\tau \langle \exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle \\
&\quad = \lambda \sum_{i\in\Lambda} \operatorname{div}\mathcal{A}(i) \langle \eta_i(\tau)\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle \\
&\qquad + \frac{\lambda^2}{2} \sum_{(i,j)\in\mathcal{E}_\Lambda} (\mathcal{A}_{(i,j)})^2 \langle \eta_i(\tau)(1-\eta_j(\tau))\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle,
\end{aligned} \tag{4.13}$$

where the second sum is over all the oriented edges. We therefore need to determine $\langle \eta_i(\tau)\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle$ to first order in λ. To do so, we can write

$$\begin{aligned}
&\partial_\tau \langle \eta_i(\tau)\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle \\
&\quad = \sum_{\substack{(k,j)\in\mathcal{E}_\Lambda \\ k,j\neq i}} (\exp(\lambda\mathcal{A}_{(k,j)}) - 1)\langle \eta_i(\tau)\eta_k(\tau)(1-\eta_j(\tau))\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle \\
&\qquad + \sum_{j\sim i} \exp(\lambda\mathcal{A}_{(j,i)})\langle \eta_j(\tau)(1-\eta_i(\tau))\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle \\
&\qquad - \langle \eta_i(\tau)(1-\eta_j(\tau))\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle,
\end{aligned}$$

where the first sum is over all the oriented edges (k, j) which do not intersect i, but the second sum is over the neighbors j of i. As for (3.11), this expression can be derived by adding the contributions of all the single moves which may occur during the infinitesimal time interval $d\tau$. Expanding to first order in λ, we get for a given τ

$$\begin{aligned}
&\partial_\tau \langle \eta_i(\tau)\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle \\
&\quad = \lambda \sum_{\substack{(k,j)\in\mathcal{E}_\Lambda \\ k,j\neq i}} \mathcal{A}_{(k,j)}\langle \eta_i(\eta_k - \eta_j)\rangle + \sum_{j\sim i}\langle \eta_j(\tau)\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle \\
&\qquad - \langle \eta_i(\tau)\exp(\lambda\mathbb{Q}^{\mathcal{A}}(\tau))\rangle + \lambda\mathcal{A}_{(j,i)}\langle \eta_j(1-\eta_i)\rangle.
\end{aligned} \tag{4.14}$$

In the periodic case, the first term in (4.14) vanishes since $\langle \eta_j(1-\eta_i)\rangle = S_{L,N}$, is independent of i, j. Hence

$$\partial_\tau \langle \eta_i(\tau)\exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle = \sum_{j;(i,j)\in\mathcal{E}_\Lambda} \langle \eta_j(\tau)\exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle - \langle \eta_i(\tau)\exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle + \lambda S_{L,N}\sum_{j\sim i}\mathcal{A}_{(j,i)}. \tag{4.15}$$

We introduce for any site k in Λ the time dependent Green's function, solution of

$$\forall i\in\Lambda,\quad \partial_\tau G^{(k)}_{\tau,i} = \Delta G^{(k)}_{\tau,i} + \delta_{i,k} = \sum_{j\sim i}(G^{(k)}_{\tau,j} - G^{(k)}_{\tau,i}) + \delta_{i,k}, \tag{4.16}$$

with the initial condition $G^{(k)}_{0,i} = 0$. Integrating (4.15), one obtains to first order in λ

$$\begin{aligned}\langle \eta_i(\tau)\exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle &= \langle \eta_i\rangle + \lambda S_{L,N}\sum_k G^{(k)}_{\tau,i}\sum_{j\sim k}\mathcal{A}_{(j,k)}\\ &= \langle \eta_i\rangle - \lambda S_{L,N}\sum_k G^{(k)}_{\tau,i}\operatorname{div}\mathcal{A}(k).\end{aligned} \tag{4.17}$$

Using (4.17) in (4.13), we get for the second order term in λ

$$\partial_\tau \langle \exp(\lambda \mathbb{Q}^{\mathcal{A}}(\tau))\rangle = \lambda^2 S_{L,N}\left(\frac{1}{2}\sum_{(i,j)\in\mathcal{E}_\Lambda}(\mathcal{A}_{(i,j)})^2 - \sum_{\substack{k\in\Lambda\\ i\in\Lambda}} G^{(k)}_{\tau,i}\operatorname{div}\mathcal{A}(k)\operatorname{div}\mathcal{A}(i)\right), \tag{4.18}$$

where the first sum is over all the oriented edges.

For the periodic square lattice $\Lambda = \{1, L\}^2$, the Green's function (4.16) is given for any $k = (k_1, k_2)$ and $i = (i_1, i_2)$ in Λ

$$G^{(k)}_{\tau,i} = \frac{1}{L^2}\left[\tau + \sum_{q_1,q_2\neq(0,0)}\frac{1-\mathrm{e}^{-(4-2\cos(q_1)-2\cos(q_2))\tau}}{4-2\cos(q_1)-2\cos(q_2)}\exp(\mathbf{i}\,q\cdot(i-k))\right],$$

with $q_1 = 2\pi\frac{m_1}{L}$, $q_2 = 2\pi\frac{m_2}{L}$ for m_1, m_2 in $\{0, L-1\}$. Using (4.18), we deduce an exact expression for the variance of the current $Q^\ell(\tau)$ through a slit ($\mathcal{A}$ is given by (4.2))

$$\langle Q^\ell(\tau)^2\rangle_c = 2S_{L,N}\int_0^\tau ds\left(\ell - \frac{1}{L^2}\sum_{q_1,q_2\neq(0,0)}\frac{(1-\mathrm{e}^{-(4-2\cos(q_1)-2\cos(q_2))s})}{2-\cos(q_1)-\cos(q_2)} \times\frac{(1-\cos(q_1))(1-\cos(q_2\ell))}{1-\cos(q_2)}\right).$$

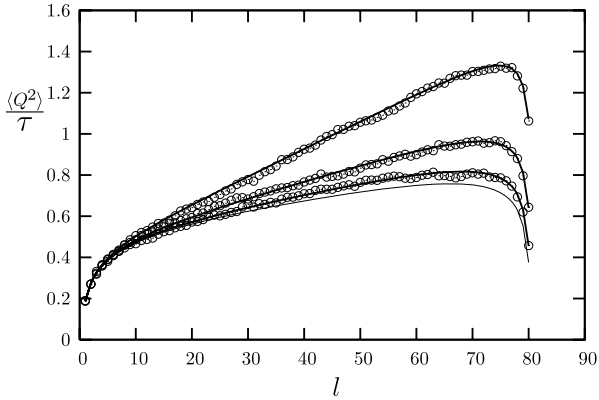

Fig. 4 $\frac{\langle Q^\ell(\tau)^2\rangle}{\tau}$ is measured versus ℓ in numerical simulations for the SSEP on a square of 80×80 sites for times $\tau = 250, 750, 2500$ (the results decrease with τ). *The continuous lines* represent the theoretical predictions (4.19) at these times, as well as the limit $\tau = \infty$ (4.11). Expression (4.19) fully agrees with the simulations

We finally get

$$\frac{\langle Q^\ell(\tau)^2\rangle_c}{\tau} = \mathrm{Var} + \frac{S_{L,N}}{\tau L^2} \sum_{q_1,q_2\neq(0,0)} \frac{(1-\mathrm{e}^{-(4-2\cos(q_1)-2\cos(q_2))\tau})}{(2-\cos(q_1)-\cos(q_2))^2} \times \frac{(1-\cos(q_1))(1-\cos(q_2\ell))}{1-\cos(q_2)}, \tag{4.19}$$

where Var, given in (4.10) or (4.11), is related to the large τ asymptotics.

4.3 Numerical Simulations

We show now the results of the simulations of the SSEP on a square lattice of size $L = 80$ with periodic boundary conditions at density $\bar{\rho} = 1/4$ (without reservoirs). The initial condition is chosen at equilibrium (i.e. the $L^2\bar{\rho}$ particles are put at random positions on the square lattice). For each simulation, we measured the flux $Q^\ell(\tau)$ through a slit of microscopic length ℓ during time τ (see Fig. 1 and (4.1)).

In Fig. 4, our data for $\frac{\langle Q^\ell(\tau)^2\rangle}{\tau}$ are compared with the predictions obtained from (4.19) for different times $\tau = 250, 750, 2500$. The simulations are averaged over 10^4 realizations. We see that unless the time is long enough, the results differ significantly from their infinite time limit (4.11). One can notice that for short times, the variance grows essentially linearly wrt ℓ as the current fluctuations are simply the sum of the (almost) independent contributions of the local current fluctuations along the slit.

In Fig. 5, the theoretical curve $\mathrm{Var} = \lim_{\tau\to\infty} \frac{\langle Q^\ell(\tau)^2\rangle}{\tau}$ computed in (4.11) is shown for several system sizes $L = 40, 80, 160, 320$. One can note that the variance of the current flowing through the whole system ($\ell = L$) is independent of L.

In Fig. 6, the same data as in Fig. 5 are shown but the horizontal axis is now ℓ/L. One can see that for large L, Var grows linearly with $\log L$ as predicted in (4.12).

5 Conclusion

In this paper, we have computed the variance of the local current for the symmetric simple exclusion process on general graphs with reservoirs (3.18) or without (3.20). In two dimensions, our exact expression leads to the asymptotics of the variance through a slit (4.12).

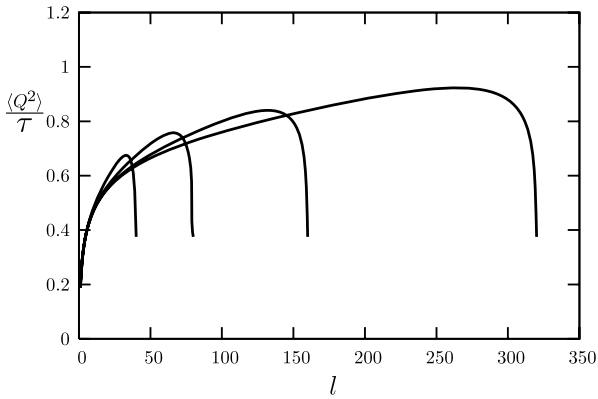

Fig. 5 Theoretical prediction (4.11) of $\lim_{\tau\to\infty} \frac{\langle Q^\ell(\tau)^2\rangle}{\tau}$ versus ℓ for $L = 40, 80, 160, 320$

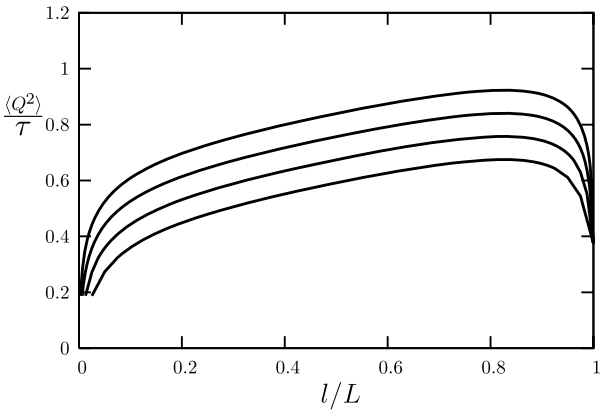

Fig. 6 Theoretical prediction (4.11) of $\lim_{\tau\to\infty} \frac{\langle Q^\ell(\tau)^2\rangle}{\tau}$ versus ℓ/L for $L = 40, 80, 160, 320$

The logarithmic dependence of the variance confirms that vortices dominate the local current fluctuations. As a consequence a fluctuation of the partial current, say the current flowing through 99% of the system, does not obey the same scaling as a fluctuation of the total current. For two dimensional diffusive models, we have also seen that the hydrodynamic large deviation theory does not catch the correct scaling of the current deviations (1.6). Finite time corrections to the variance were also computed (4.19) and compared to numerical data (Fig. 4). Finally the fluctuation relation (1.4) for partial currents is discussed in Appendix A.

It would be interesting to investigate the scaling of partial current deviations in higher dimensions. Another challenging issue is the computation of the full large deviation functional for partial currents.

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Appendix A: The Fluctuation Theorem and Partial Currents

For the total current, the fluctuation theorem (1.4) holds (in any dimension)

$$G_{L,L}(J;\rho_a,\rho_b) - G_{L,L}(-J;\rho_a,\rho_b) = J[\log z(\rho_b) - \log z(\rho_a)]. \tag{A.1}$$

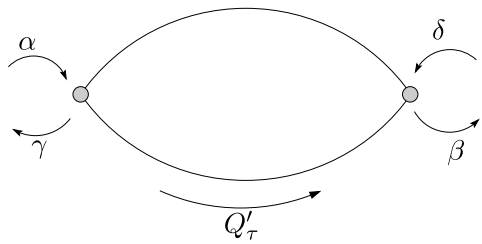

Fig. 7 A reservoir at density ρ_a (resp ρ_b) is acting on the *left* (resp *right*) site by creating particles at rate α (resp δ) and annihilating particles at rate γ (resp β). We consider the large deviations of the current Q'_τ flowing through the lower edge

This fluctuation relation is based on a global symmetry: the fluctuation to produce the current $-J$ is simply the time reversal of the fluctuation to produce the current J. One may wonder how this generalizes to the function $G_{L,\ell}$. In this appendix, we show by considering a very simplified model that the fluctuation relation (A.1) is in general not satisfied for partial current deviations.

We consider the SSEP with two sites $\{1, 2\}$ connected to reservoirs. At site 1, creation (resp annihilation) occurs at rate α (resp γ) and at site 2, creation (resp annihilation) occurs at rate δ (resp β) (see Fig. 7). The exchanges between sites $\{1, 2\}$ obey the usual exclusion rule, but they can occur through two edges with rate 1. On the one hand the model behaves like a SSEP with exchange rate 2. Thus the total current flowing from site 1 to site 2 obeys the fluctuation relation (A.1) (with $\rho_a = \frac{\alpha}{\alpha+\gamma}$ and $\rho_b = \frac{\delta}{\delta+\beta}$). On the other hand one can also consider a current deviation J through one of the two edges. Heuristically one can see that the system is going to use different strategies to produce a current J or $-J$. Imagine the extreme case with only creation at site 1 and annihilation at site 2 ($\gamma = 0$ and $\delta = 0$). For the total current, there is no way of producing a negative flux and the relation (A.1) is degenerate: $\log z(\rho_b) - \log z(\rho_a) = -\infty$. On the other hand, a negative current can be achieved through the lower edge by letting a single particle cross the lower edge from site 2 to site 1 and then use the upper edge to go back to site 2. This latter mechanism mimics the vortices discussed in Sect. 2. Thus, the fluctuations to produce current deviations J or $-J$ are not related by time reversal. In general, both mechanisms (total current deviation and local vortices) combine and there is no reason to expect a symmetry such as (A.1).

We analyze now the toy model analytically. We define Q'_τ as the integrated current flowing through the lower edge during the time interval $[0, \tau]$ (see Fig. 7). Instead of trying to check an expression like (A.1) for the large deviation function we look for a symmetry at the level of its Legendre transform. As in [10], one knows that

$$\forall \lambda, \quad \lim_{\tau\to\infty} \frac{1}{\tau} \log\langle \exp(\lambda Q'_\tau)\rangle = \mu(\lambda),$$

where $\mu(\lambda)$ is the largest eigenvalue of the operator

$$L_\lambda = \begin{pmatrix} -\alpha-\delta & \gamma & \beta & 0 \\ \alpha & -\gamma-\delta-2 & 1+e^{-\lambda} & \beta \\ \delta & 1+e^{\lambda} & -\alpha-\beta-2 & \gamma \\ 0 & \delta & \alpha & -\gamma-\beta \end{pmatrix}.$$

The fluctuation relation (A.1) would say that there exists a constant E such that

$$\forall \lambda, \quad \mu(-\lambda - 2E) = \mu(\lambda). \tag{A.2}$$

In order to prove that the previous relation does not hold, we consider for simplicity the case $\alpha = 2$, $\gamma = 1$ and $\delta = 1$, $\beta = 2$. Then the characteristic polynomial of L_λ is

$$P(u) = (3+u)(16+u(44+u(13+u)) - 2(8+u)\cosh[\lambda] - 6\sinh[\lambda]).$$

For (A.2) to be satisfied, $\mu(\lambda)$ should be a root of this polynomial and of the polynomial associated to $L_{-\lambda-2E}$. This implies that

$$2(8+\mu(\lambda))\cosh[\lambda] + 6\sinh[\lambda] = 2(8+\mu(\lambda))\cosh[-\lambda-2E] + 6\sinh[-\lambda-2E]$$

leading to

$$\forall\lambda, \quad \mu(\lambda) = -3\coth[E] - 8 \quad \text{or} \quad \mu(\lambda) = -3.$$

As $\mu(\lambda)$ cannot be independent of λ, we obtained a contradiction. Thus the fluctuation relation (A.2) does not hold in this toy model.

For the total current a similar calculation shows that $\mu(\lambda)$ is the root of

$$Q(u) = (3+u)(20+u(6+u)(7+u) - 20\cosh[\lambda] - 12\sinh[\lambda]).$$

This is invariant under the symmetry $\lambda \to -2\log 2 - \lambda$, implying that (A.2) is satisfied.

Appendix B

In this appendix we derive the large L, ℓ expression (4.12) of the variance (4.11).

Define I_N and J_N by

$$I_N = \sum_{n=1}^{N} \frac{1}{n^2+b^2} \quad \text{and} \quad J_N = \int_0^N \frac{dx}{b^2+x^2}.$$

One has for large N

$$\begin{aligned} I_N &= J_N + \pi^2\left[\frac{1}{2\pi b\tanh(\pi b)} - \frac{1}{2\pi^2 b^2} - \frac{1}{2\pi b}\right] + o(1) \\ &= J_N + \pi^2\left[\frac{1}{\pi b[\exp(2\pi b)-1]} - \frac{1}{2\pi^2 b^2}\right] + o(1). \end{aligned} \tag{B.1}$$

Recall also that

$$\int_0^1 \frac{dx}{2-B-\cos(2\pi x)} = \frac{1}{\sqrt{(1-B)(3-B)}}. \tag{B.2}$$

From (B.1, B.2), one can show, by taking $b^2 = L^2(1-\cos q_2)/(2\pi^2)$, that for $0 < q_2 < 2\pi$

$$\begin{aligned} \frac{1}{L}\sum_{n_1=0}^{L-1} \frac{1}{2-\cos q_2 - \cos\frac{2\pi n_1}{L}} &= \frac{1}{\sqrt{(1-\cos q_2)(3-\cos q_2)}} \\ &\quad + \frac{1}{\sin(\frac{q_2}{2})(\exp[2L\sin(\frac{q_2}{2})]-1)} + o(1). \end{aligned} \tag{B.3}$$

The main contribution to the difference between the sum in (B.3) and integral (B.2) is given by the terms with n_1 close to 0 or to L. In both cases $(1-\cos\frac{2\pi n_1}{L})$ can be approximated by its second order expansion and the last term in (B.3) is obtained thanks to (B.1).

(B.3) can be rewritten as

$$\frac{1}{L}\sum_{n_1=0}^{L-1}\frac{1}{2-\cos q_2-\cos\frac{2\pi n_1}{L}}$$
$$=\frac{1-\sqrt{\frac{3-\cos q_2}{2}}}{\sqrt{(1-\cos q_2)(3-\cos q_2)}}+\frac{1}{2\sin\frac{q_2}{2}}+\frac{1}{\sin(\frac{q_2}{2})(\exp[2L\sin(\frac{q_2}{2})]-1)}+o(1).$$

One can then perform the sum over q_2. For large L, the first term becomes an integral

$$\int_0^1 dx\frac{1-\sqrt{\frac{3-\cos(2\pi x)}{2}}}{\sqrt{[1-\cos(2\pi x)][3-\cos(2\pi x)]}}=\frac{1}{2\pi}\int_0^\pi d\phi\frac{1-\sqrt{1+\sin^2\phi}}{\sin\phi\,\sqrt{1+\sin^2\phi}}=-\frac{\log 2}{2\pi}.$$

For large L one can also show that

$$\frac{1}{L}\sum_{n=1}^{L-1}\frac{1}{2\sin\frac{n\pi}{L}}\simeq\frac{1}{\pi}\left[\log L+\log\left(\frac{2}{\pi}\right)+\gamma_E\right]+o(1).$$

For large l and L with $l=Lh$

$$\frac{1}{L}\sum_{n=1}^{L-1}\frac{\cos(\frac{2n\pi}{L}l)}{2\sin\frac{n\pi}{L}}\simeq-\frac{1}{2\pi}\log(2-2\cos(2\pi h))=-\frac{1}{\pi}\log(2\sin(\pi h)).$$

There is also the identity

$$\sum_{n=1}^{\infty}\frac{2[1-\cos(2\pi nh)]}{n\pi(e^{2n\pi}-1)}=\frac{1}{\pi}\sum_{m=1}^{\infty}\log\left(1+\frac{\sin^2(\pi h)}{\sinh^2(\pi m)}\right).$$

Putting everything together one gets that

$$\frac{1}{L^2}\sum_{n_1=0}^{L-1}\sum_{n_2=1}^{L-1}\frac{1-\cos(q_2 l)}{2-\cos q_2-\cos q_1}$$
$$\simeq\frac{1}{\pi}\left[\log L+\log\left(\frac{\sin(\pi h)}{\pi}\right)+\frac{3\log 2}{2}+\gamma_E+\sum_{m\geq 1}\log\left(1+\frac{\sin^2(\pi h)}{\sinh^2(\pi m)}\right)\right].$$

Note that for h small, i.e for $1\ll l\ll L$, one recovers a well known expression (see [16], p. 198).

References

1. Andrieux, D., Gaspard, P.: Network and thermodynamic conditions for a single macroscopic current fluctuation theorem. Comptes Rendus Phys. **8**(5–6), 579–590 (2007)

2. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Nonequilibrium current fluctuations in stochastic lattice gases. J. Stat. Phys. **123**(2), 237–276 (2006)
3. Bertini, L., De Sole, A., Gabrielli, D., Jona–Lasinio, G., Landim, C.: Current fluctuations in stochastic lattice gases. Phys. Rev. Lett. **94**, 030601 (2005)
4. Bertini, L., De Sole, A., Gabrielli, D., Jona–Lasinio, G., Landim, C.: Large deviations of the empirical current in interacting particle systems. Theory Probab. Appl. **51**(1), 2–27 (2007)
5. Bodineau, T., Derrida, B.: Current fluctuations in nonequilibrium diffusive systems: an additivity principle. Phys. Rev. Lett. **92**, 180601 (2004)
6. Bodineau, T., Derrida, B.: Distribution of current in nonequilibrium diffusive systems and phase transitions. Phys. Rev. E (3) **72**(6), 066110 (2005)
7. Bodineau, T., Derrida, B.: Cumulants and large deviations of the current through non-equilibrium steady states. Comptes Rendus Phys. **8**(5–6), 540–555 (2007)
8. Bonetto, F., Lebowitz, J.: Thermodynamic entropy production fluctuation in a two dimensional shear flow model. Phys. Rev. E **64**, 056129 (2001)
9. Ciliberto, S., Laroche, C.: An experimental test of the Gallavotti-Cohen fluctuation theorem. J. Phys. IV **8**, 215 (1998)
10. Derrida, B., Douçot, B., Roche, P.-E.: Current fluctuations in the one dimensional symmetric exclusion process with open boundaries. J. Stat. Phys. **115**, 717–748 (2004)
11. Donsker, M., Varadhan, S.R.S.: Large deviations from a hydrodynamic scaling limit. Commun. Pure Appl. Math. **42**(3), 243–270 (1989)
12. Evans, D.J., Cohen, E.G.D., Morriss, G.P.: Probability of second law violations in shearing steady states. Phys. Rev. Lett. **71**, 2401 (1993)
13. Evans, D.J., Searles, D.J.: The fluctuation theorem. Adv. Phys. **51**, 1529–1585 (2002)
14. Gallavotti, G., Cohen, E.D.G.: Dynamical ensembles in stationary states. J. Stat. Phys. **80**, 931–970 (1995)
15. Gallavotti, G.: Entropy production in nonequilibrium thermodynamics: a point of view. Chaos **14**, 680–690 (2004)
16. Itzykson, C., Drouffe, J.M.: Statistical Field Theory, vol. 1. From Brownian Motion to Renormalization and Lattice Gauge Theory. Cambridge Monographs on Mathematical Physics (1989)
17. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Springer, Berlin (1999)
18. Kipnis, C., Olla, S., Varadhan, S.: Hydrodynamics and large deviations for simple exclusion processes. Commun. Pure Appl. Math. **42**, 115–137 (1989)
19. Kurchan, J.: Fluctuation theorem for stochastic dynamics. J. Phys. A **31**, 3719 (1998)
20. Lebowitz, J.L., Spohn, H.: A Gallavotti-Cohen type symmetry in the large deviation functional for stochastic dynamics. J. Stat. Phys. **95**, 333–366 (1999)
21. Maes, C.: The fluctuation theorem as a Gibbs property. J. Stat. Phys. **95**, 367–392 (1999)
22. Spohn, H.: Large Scale Dynamics of Interacting Particles. Springer, Berlin (1991)